

Weak Factorization Systems and Fibrewise Regular Injectivity for Actions of Po-monoids on Posets

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Abstract

Let S be a pomonoid. In this paper, $\mathbf{Pos}\text{-}S$, the category of S -posets and S -poset maps, is considered. One of the main aims of this paper is to draw attention to the notion of weak factorization systems in $\mathbf{Pos}\text{-}S$. We show that if the identity element of S is the bottom element, then $(\mathcal{C}_{\mathcal{D}}, \mathcal{E}_{\mathcal{S}})$ is a weak factorization system in $\mathbf{Pos}\text{-}S$, where $\mathcal{C}_{\mathcal{D}}$ and $\mathcal{E}_{\mathcal{S}}$ are the class of down-closed embedding S -poset maps and the class of all split S -poset epimorphisms, respectively. Among other things, we use a fibrewise notion of complete posets in the category $\mathbf{Pos}\text{-}S/B$ under a particular case where B has trivial action. We get a necessary condition for regular injective objects in $\mathbf{Pos}\text{-}S/B$. Finally, we characterize them under a spacial case, where S

is, a pogroup and conclude (Emb, Top) is a weak factorization system in **Pos**- S .

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1 Introduction

A comma category (a special case being a slice category) is a construction in category theory. It provides another way of looking at morphisms: instead of simply relating objects of a category to one another, morphisms become objects in their own right. This notion was introduced in 1963 by F. W. Lawvere, although the technique did not become generally known until many years later.

Injective objects with respect to a class \mathcal{H} of morphisms have been investigated for a long time in various categories. Recently, injective objects in slice categories (\mathbf{C}/B) have been investigated in detail (see [1, 6]), especially in relationship with weak factorization systems, a concept used in homotopy theory, in particular for model categories. More precisely, \mathcal{H} -injective objects in \mathbf{C}/B , for any B in \mathbf{C} , form the right part of a weak factorization system that has morphisms of \mathcal{H} as the left part (see [1, 2]).

In this paper, the notion of weak factorization system in **Pos**- S is investigated. After some introductory notions in section 1, we introduce in section 2, the notion of weak factorization system and state some related basic theorems. Also, we give the guarantee about the existence of (Emb, Emb^\square) as a weak factorization system in **Pos**- S , where Emb is the class of all order-embeddings. We then find that every Emb -injective object in **Pos**- S/B is split epimorphism. In section 3, we continue studying Emb -injectivity using a fibrewise notion of complete posets in the category **Pos**- S/B under a particular case where B has trivial action.

For the rest of this section, we give some preliminaries which we will need in the sequel.

Given a category \mathbf{C} and an object B of \mathbf{C} , one can construct the *slice category* \mathbf{C}/B (read: \mathbf{C} over B): objects of \mathbf{C}/B are morphisms of \mathbf{C} with codomain B , and morphisms in \mathbf{C}/B from one such object $f : A \rightarrow B$ to another $g : C \rightarrow B$ are commutative triangles in \mathbf{C}

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ & \searrow f & \swarrow g \\ & B & \end{array}$$

i.e, $gh = f$. The composition in \mathbf{C}/B is defined from the composition in \mathbf{C} , in the obvious way (paste triangles side by side).

Let \mathbf{C} be a category and \mathcal{H} a class of its morphisms. An object I of \mathbf{C} is called \mathcal{H} -*injective* if for each \mathcal{H} -morphism $h : U \rightarrow V$ and morphism $u : U \rightarrow I$ there exists a morphism $s : V \rightarrow I$ such that $sh = u$. That is, the following diagram is commutative:

$$\begin{array}{ccc} U & \xrightarrow{u} & I \\ h \downarrow & \nearrow s & \\ V & & \end{array}$$

In particular, in the slice category \mathbf{C}/B , this means that, $f : X \rightarrow B$ is \mathcal{H} -injective if, for any commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ h \downarrow & & \downarrow f \\ V & \xrightarrow{v} & B \end{array}$$

with $h \in \mathcal{H}$, there exists an arrow $s : V \rightarrow X$ such that $sh = u$ and $fs = v$.

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ h \downarrow & \nearrow s & \downarrow f \\ V & \xrightarrow{v} & B \end{array}$$

The category \mathbf{C} is said to have enough \mathcal{H} -injectives if for every object A of \mathbf{C} there exists a morphism $A \rightarrow C$ in \mathcal{H} where C is an \mathcal{H} -injective object in \mathbf{C} .

Let S be a monoid with identity 1. A (*right*) S -act or S -set is a set A equipped with an action $\mu : A \times S \rightarrow A$, $(a, s) \mapsto as$, such

that $a1 = a$ and $a(st) = (as)t$, for all $a \in A$ and $s, t \in S$. Let **Act- S** denote the category of all S -acts with action-preserving maps or S -maps. Clearly S itself is an S -act with its operation as the action. For instance, take any monoid S and a non-empty set A . Then A becomes a right S -act by defining $as = a$ for all $a \in A$, $s \in S$, we call that A an S -act with *trivial action* (see [10] or [11]).

- A poset is said to be *complete* if each of its subsets has an infimum and a supremum.

Recall that a *pomonoid* is a monoid with a partial order \leq which is compatible with the monoid operation: for $s, t, s', t' \in S$, $s \leq t$, $s' \leq t'$ imply $ss' \leq tt'$. Now, let S be a pomonoid. A (*right*) S -*poset* is a poset A which is also an S -act whose action $\mu : A \times S \rightarrow A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order. The category of all S -posets with action preserving monotone maps between them is denoted by **Pos- S** . Clearly S itself is an S -poset with its operation as the action. Also, if B is a non-empty subposet of A , then B is called a *sub S -poset* of A if $bs \in B$ for all $s \in S$ and $b \in B$. Throughout this paper we deal with the pomonoid S and the category **Pos- S** , unless otherwise stated. For more information on S -posets see [5] or [8].

2 Weak Factorization System

The concept of weak factorization systems plays an important role in the theory of model categories. Formally, this notion generalizes factorization systems by weakening the unique diagonalization property to the diagonalization property without uniqueness. However, the basic examples of weak factorization systems are fundamentally different from the basic examples of factorization systems.

Now, we introduce from [1] the notion which we deal with in the paper.

Notation: We denote by \square the relation *diagonalization property* on the class of all morphisms of a category **C**: given morphisms $l : A \rightarrow B$ and $r : C \rightarrow D$ then

$$l \square r$$

means that in every commutative square

$$\begin{array}{ccc} A & \longrightarrow & C \\ l \downarrow & \nearrow d & \downarrow r \\ B & \longrightarrow & D \end{array}$$

there exists a diagonal $d : B \rightarrow C$ rendering both triangles commutative. In this case l is also said to have the *left lifting property* with respect to r (and r to have the *right lifting property* with respect to l).

Let \mathcal{H} be a class of morphisms. We denote by

$$\mathcal{H}^\square = \{r \mid r \text{ has the right lifting property with respect to each } l \in \mathcal{H}\}$$

and

$${}^\square\mathcal{H} = \{l \mid l \text{ has the left lifting property with respect to each } r \in \mathcal{H}\}.$$

Let \mathcal{H}_B be the class of those morphisms in \mathbf{C}/B whose underlying morphism in \mathbf{C} lies in \mathcal{H} . Now, $r : A \rightarrow B \in \mathcal{H}^\square$ if and only if r is an \mathcal{H}_B -injective object in \mathbf{C}/B . Dually, all morphisms in ${}^\square\mathcal{H}$ are characterized by a projectivity condition in \mathcal{H}_B .

Recall from [1] that a *weak factorization system* in a category is a pair $(\mathcal{L}, \mathcal{R})$ of morphism classes such that

- (1) every morphism has a factorization as an \mathcal{L} -morphism followed by an \mathcal{R} -morphism.
- (2) $\mathcal{R} = \mathcal{L}^\square$ and $\mathcal{L} = {}^\square\mathcal{H}$.

Remark 2.1. If we replace “ \square ” by “ \perp ” where “ \perp ” is defined via the *unique diagonalization property* (i.e., by insisting that there exists precisely one diagonal), we arrive at the familiar notion of a factorization system in a category. Factorization systems are weak factorization systems. For instance, let \mathcal{E} be the class of all S -poset epimorphisms. Then, by Theorem 1 of [5] one can easily see that $(\mathcal{E}, \text{Emb})$ in $\mathbf{Pos}\text{-}S$ is a factorization system.

Now, consider a functor $G : \mathcal{A} \rightarrow \mathcal{X}$. Recall from [1] that a source $(A \rightarrow A_i)_{i \in I}$ in \mathcal{A} is called G -initial provided that for each source $(g_i : B \rightarrow A_i)_{i \in I}$ in \mathcal{A} and each \mathcal{X} -morphism $h : GB \rightarrow GA$ with $Gg_i = Gf_i h$ for each $i \in I$, there exists a unique \mathcal{A} -morphism $\bar{h} : B \rightarrow$

A in \mathcal{A} with $G\bar{h} = h$ and $g_i = f_i\bar{h}$ for each $i \in I$.

Also, a source $(A \rightarrow A_i)_{i \in I}$ lifts a G -structured source $(f_i : X \rightarrow GA_i)_{i \in I}$ provided that $G\bar{f}_i = f_i$ for each $i \in I$.

Definition 2.2. A functor $G : \mathcal{A} \rightarrow \mathcal{X}$ in the category **Cat** (of all categories and functors) is topological if every G -structured source $(X \rightarrow G(A_i))_{i \in I}$ has a unique G -initial lift $(A \rightarrow A_i)_{i \in I}$.

Example 2.3. (1) In the category **Set** (of all sets and functions between them) pair $(Mono, Epi)$ is a weak factorization system. But $(Epi, Mono)$ is a factorization system in this category and also in other categories, where $Mono$ is the class of all monomorphisms and Epi is the class of all epimorphisms in **Set**.

(2) The pair $(Full, Top)$ is a weak factorization system in the category **Cat**, where $Full$ is the class of those morphisms in **Cat** that are full and Top is the class of those morphisms in **Cat** that are topological.

(3) In the category **Pos** of all posets with monotone maps, the pair (Emb, Top) is a weak factorization system, where Emb is the class of all order-embeddings; that is, maps $f : A \rightarrow B$ for which $f(a) \leq f(a')$ if and only if $a \leq a'$, for all $a, a' \in A$ and Top is the class of all topological monotone maps. For more details of proof see [1].

We record the following two results from [1], that will be used in the sequel.

Proposition 2.4. Let \mathbf{C} be a category and \mathcal{H} a class of morphisms closed under retracts in slice category \mathbf{C}/B for all objects B of \mathbf{C} . Then the following conditions are equivalent:

- (1) $(\mathcal{H}, \mathcal{H}^\square)$ is a weak factorization system.
- (2) \mathbf{C}/B has enough \mathcal{H}_B -injectives.

Proposition 2.5. Let \mathbf{C} be a category. Then $(\mathcal{L}, \mathcal{R})$ is a weak factorization system if and only if

- (1) Any morphism $h \in \mathbf{C}$ has a factorization $h = gf$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.
- (2) For all $f \in \mathcal{L}$ and $g \in \mathcal{R}$, f has the left lifting property with respect to g .
- (3) If $f : A \rightarrow B$ and $f' : X \rightarrow Y$ are such that there exist morphisms

$\alpha : B \rightarrow Y$ and $\beta : A \rightarrow X$ then

- (a) If $\alpha f \in \mathcal{L}$ and if α is a split monomorphism then $f \in \mathcal{L}$.
- (b) If $f'\beta \in \mathcal{R}$ and if β is split epimorphism then $f' \in \mathcal{R}$

If $f : A \rightarrow B$ and $g : A \rightarrow C$ are morphisms in a category \mathbf{C} such that there exist morphisms $\alpha : C \rightarrow B$ and $\beta : B \rightarrow C$ with $\beta\alpha = 1_C$, $\alpha g = f$ and $\beta f = g$ then we say that g is a *retract* of f . In categorical terms, g is a retract of f in the coslice category A/\mathbf{C} .

Notice that in 3(a) above, f is a retract of αf and that all retracts can be written in this way. So this result is simply saying that \mathcal{L} is closed under retracts. Similarly 3(b) is equivalent to \mathcal{R} being closed under retracts.

Recently, Bailey and Renshaw in [2], provide a number of examples of weak factorization systems for S -acts such as the following theorem. But first we need a definition.

Definition 2.6. We say that an S -act (S -poset) monomorphism $f : X \rightarrow Y$ is *unitary* if $y \in \text{im}(f)$ whenever $ys \in \text{im}(f)$ and $s \in S$. Clearly this is equivalent to saying that there exists an S -act (S -poset) Z such that $Y \cong X \dot{\cup} Z$ or in other words, $\text{im}(f)$ is a direct summand of Y .

Theorem 2.7. Let S be a monoid and let \mathcal{U} be the class of all unitary S -monomorphisms and \mathcal{E}_S be the class of all split S -act epimorphisms. Then $(\mathcal{U}, \mathcal{E}_S)$ is a weak factorization system in $\mathbf{Act}\text{-}S$.

2.1 Weak Factorization Systems via Down-closed Embeddings

Now, consider Emb as the class of all embeddings of S -posets. We try to provide a weak factorization system for $\mathbf{Pos}\text{-}S$ with Emb as the left part. In this subsection, we consider down-closed embeddings, as a subclass of Emb , and find a weak factorization system in $\mathbf{Pos}\text{-}S$ with a condition on pomonoid S .

Definition 2.8 ([12]). A possibly empty sub S -poset A of an S -poset B is said to be *down-closed* in B if for each $a \in A$ and $b \in B$ with $b \leq a$

we have $b \in A$. By a *down-closed embedding*, we mean an embedding $f : A \rightarrow B$ such that $f(A)$ is a down-closed sub S -poset of B .

Now, we prove the following crucial lemma in the category **Pos**- S .

Lemma 2.9. *Let S be a pomonoid whose identity element e is the bottom element and $f : X \rightarrow Y$ be an S -poset map. If f is a down-closed embedding then $\text{im}(f)$ is a direct summand of Y .*

Proof. First we show that if $y \in \text{im}(f)$ whenever $ys \in \text{im}(f)$ and $s \in S$. In fact, by hypothesis we have $e \leq s$ and we get $y \leq ys$, for every $y \in Y$ and $s \in S$. Now, as $\text{im}(f)$ is down-closed and $ys \in \text{im}(f)$ then $y \in \text{im}(f)$. Second, it is easy to see that the above property of f is equivalent to saying that there exists an S -poset Z (put $Z = Y \setminus \text{im}(f)$) such that $Y \cong X \dot{\cup} Z$ or in other words, $\text{im}(f)$ is direct summand of Y . \square

Let $\mathcal{C}_{\mathcal{D}}$ denotes the class of down closed embedding S -poset maps and let $\mathcal{E}_{\mathcal{S}}$ denotes the class of all split S -poset epimorphisms. We shall provide a weak factorization system for **Pos**- S by these two classes. In other words;

Theorem 2.10. *Let S be a pomonoid whose identity is the bottom element and $\mathcal{C}_{\mathcal{D}}$ and $\mathcal{E}_{\mathcal{S}}$ as above. Then $(\mathcal{C}_{\mathcal{D}}, \mathcal{E}_{\mathcal{S}})$ is a weak factorization system in **Pos**- S .*

Proof. We must show all conditions of Proposition 2.5. For (1), if $f : X \rightarrow Y$ is an S -poset map then we define the split epimorphism $\bar{f} : X \dot{\cup} Y \rightarrow Y$ by $\bar{f}(x) = f(x)$, $\bar{f}(y) = y$ for all $x \in X$, $y \in Y$. Let $i : X \rightarrow X \dot{\cup} Y$ be the inclusion, it is easy to see that i is a down-closed embedding. Now, we have $f = \bar{f}i$ and $i \in \mathcal{C}_{\mathcal{D}}$, $\bar{f} \in \mathcal{E}_{\mathcal{S}}$.

For condition (2), consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & C \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{v} & D \end{array}$$

with $f \in \mathcal{C}_{\mathcal{D}}$ and $g \in \mathcal{E}_{\mathcal{S}}$. In view of Lemma 2.9, without loss of generality we may assume that f is of the form $f : X \rightarrow X \dot{\cup} Z$. Then

there exists $h : D \rightarrow C$ with $gh = 1_D$ and so we can define an S -poset map $k : X \dot{\cup} Z \rightarrow C$ by $k|_X = u$, $k|_Z = hv|_Z$ with the required property. Finally, suppose that f, f', α, β are as in condition (3) of Proposition 2.5. If $f'\beta$ is a split epimorphism then clearly so is f' . If $g : Y \rightarrow A$ is such that $(f'\beta)g = 1_Y$ then $f' \in \mathcal{E}_S$ with splitting morphism βg . Suppose that $\alpha f \in \mathcal{C}_D$ and if α is a split monomorphism. Since αf is an embedding, f is also. Now, we show that $f(A)$ is a down-closed sub S -poset of Y . Given $f(a) \in \text{im}(f)$ for some $a \in A$ and $b \leq f(a)$. Then $\alpha(b) \leq \alpha f(a)$. As αf is a down-closed embedding we have $\alpha(b) \in \text{im}(\alpha f)$. In fact, $b \in \text{im}(f)$ as α is a monomorphisms (exactly one to one (see[5])). Consequently, $f \in \mathcal{C}_D$. \square

Recall that each poset can be embedded (via an order-embedding) into a complete poset, called the Dedekind-MacNeille completion. In fact, given a poset P , its MacNeille completion is the poset \bar{P} consisting of all subsets A of P for which $LU(A) = A$, where

$$U(A) = \{x \in P : x \geq a, \forall a \in A\}$$

and

$$LU(A) = \{y \in P : y \leq x, \forall x \in U(A)\},$$

and the embedding $\downarrow(-) : P \rightarrow \bar{P}$ is given by

$$a \mapsto \downarrow(a) = \{x \in P : x \leq a\}$$

(see [3]).

Notice that in the category $\mathbf{Pos}\text{-}S/B$, regular monomorphisms correspond to regular monomorphisms in $\mathbf{Pos}\text{-}S$ and these are exactly order-embeddings (in $\mathbf{Pos}\text{-}S$) (see [5, 9]). We state the following theorem which gives us enough *Emb*-injectivity property in $\mathbf{Pos}\text{-}S/B$. For details of the proof see [9].

Theorem 2.11. *For an arbitrary S -poset B , the category $\mathbf{Pos}\text{-}S/B$ has enough regular injectives. More precisely, each object $f : A \rightarrow B$ in $\mathbf{Pos}\text{-}S/B$ can be regularly embedded into a regular injective object $\pi_B^{\bar{A}^{(S)}} : \bar{A}^{(S)} \times B \rightarrow B$ in $\mathbf{Pos}\text{-}S/B$ in which $\bar{A}^{(S)}$ is the set of all monotone maps from S into \bar{A} , with pointwise order and the action*

is given by $(fs)(t) = f(st)$ for $s, t \in S$ and $f \in \bar{A}^{(S)}$ and $\pi_B^{\bar{A}^{(S)}} : \bar{A}^{(S)} \times B \rightarrow B$ is the second projection.

It is easy to show that the class Emb closed under retracts in $\mathbf{Pos}\text{-}S/B$. So by the Proposition 2.4 and above theorem, we can say that (Emb, Emb^\square) is a weak factorization system for $\mathbf{Pos}\text{-}S$. This implies that Emb is saturated (this means, every class in a category is closed under pushouts, transfinite compositions and retracts (see [2])).

Up to now, we can not succeed to determine if there is a class \mathcal{R} such that (Emb, \mathcal{R}) is a weak factorization system. However we do have:

Proposition 2.12. *Let S be a pomonoid. Suppose (Emb, \mathcal{R}) is a weak factorization system for $\mathbf{Pos}\text{-}S$. Then $\mathcal{R} \subseteq \mathcal{E}_S$.*

Proof. Let $f : A \rightarrow B \in \mathcal{R}$ and consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\text{id}} & A \\ i \downarrow & & \downarrow f \\ A \dot{\cup} B & \xrightarrow{\bar{f}} & B \end{array}$$

where i is the inclusion map and where $\bar{f}|_A = f$, $\bar{f}|_B = \text{id}$. By assumption we have $\mathcal{R} = Emb^\square$, so there exists $h : A \dot{\cup} B \rightarrow A$ such that $fh = \bar{f}$ and $hi = \text{id}$. This implies that f is split epimorphism. Thus $\mathcal{R} \subseteq \mathcal{E}_S$. \square

3 Fibrewise Regular Injectivity of S -Poset Maps

In the last section, we deduced that every Emb -injective objects in $\mathbf{Pos}\text{-}S/B$ is a split epimorphism (see Proposition 2.12). In this section, we are going to characterize them using a fibrewise notion of complete posets.

We recall that in the category \mathbf{Pos} of partially ordered sets and monotone maps, a monotone map can be characterized as follows.

Theorem 3.1 ([13]). *A monotone map $f : X \rightarrow B$ is Emb-injective in \mathbf{Pos}/B if and only if it satisfies the following conditions:*

- (I) $f^{-1}(b)$ is complete poset, for every $b \in B$.
- (II) f is a fibration (that is, for every $x \in X$ and $b \in B$ with $f(x) \leq b$, $\{x' \in f^{-1}(b) \mid x \leq x'\}$ has a minimum element) and a cofibration (=dual of fibration).

The category of $\mathbf{Pos}\text{-}S$ is cartesian closed (see [5]). So by Theorem 1.2 from [6] we have:

Theorem 3.2. *Let S be a pomonoid. Then $f : X \rightarrow B$ is a regular injective object in $\mathbf{Pos}\text{-}S/B$ if and only if the following two conditions are satisfied:*

- (1) $\langle 1_X, f \rangle : f \rightarrow \pi_B^X$ is a section in $\mathbf{Pos}\text{-}S/B$ where $\pi_B^X : X \times B \rightarrow B$ is the second projection.
- (2) The object $S_B(f)$ of sections of f is a regular injective object in $\mathbf{Pos}\text{-}S$.

Now, we supply a partial answer to the characterization of regular injectivity in the category $\mathbf{Pos}\text{-}S/B$ in a special case, when the S -poset B has the trivial action.

Remark 3.3 ([9]). For a pomonoid S , we know that $\mathbf{Pos}\text{-}S$ is cartesian closed. Indeed, given two S -posets A and B the exponential B^A is given by $B^A = \text{Hom}(S \times A, B)$, the set of all S -poset maps from the product S -poset $S \times A$ to B . Note that the action on $S \times A$ operates on both components. This set is an S -poset, with pointwise order and the action is given by $(fs)(t, a) = f(st, a)$ (see [5, 11]). Now, given $f : X \rightarrow B$ in $\mathbf{Pos}\text{-}S/B$ we have

$$S_B(f) = \{h \in \text{Hom}(S \times B, X) \mid fh = \pi_B^S\}.$$

Also, we consider the following embedding induced by fibres of f :

$$m : S(f) \hookrightarrow \prod_{b \in B} f^{-1}(b) \quad \text{with} \quad m(h) = (h(b, s))_{s \in S, b \in B}$$

and recall the following result from [9].

Proposition 3.4. *Let S be a pomonoid. If $f : X \rightarrow B$ is a regular injective object in the category $\mathbf{Pos}\text{-}S/B$, then:*

- (1) $\langle 1_X, f \rangle : X \rightarrow X \times B$ is a section in $\mathbf{Pos}\text{-}S/B$.
- (2) For every $b \in B$, the sub S -poset $f^{-1}(b)$ of X is regular injective object in $\mathbf{Pos}\text{-}S$, so it is a complete poset.

In this section, we are going to give a new characterization of injective objects in $\mathbf{Pos}\text{-}S/B$ that removes condition (1) of the above proposition.

Proposition 3.5. *Let S be a pomonoid and $f : X \rightarrow B$ be a S -poset map. If $\langle 1_X, f \rangle : X \rightarrow X \times B$ is a section in $\mathbf{Pos}\text{-}S/B$ then for $x \in X$ and $b \in B$ with $f(x) \leq b$, $\{x' \in f^{-1}(b) \mid x \leq x'\}$ has a minimum element.*

Proof. Let $r : X \times B \rightarrow X$ be a retraction of $\langle 1_X, f \rangle$ over B . For $x \in X$ and $b \in B$ with $f(x) \leq b$, let $r(x, b) = x_b$ so we have

$$x = r(x, f(x)) \leq r(x, b) = x_b$$

Also, take x' in $f^{-1}(b)$ with $x \leq x'$ then

$$x_b = r(x, b) \leq r(x', b) = x'$$

This means that x_b is minimum in $\{x' \in f^{-1}(b) \mid x \leq x'\}$. □

Corollary 3.6. *Let S be a pomonoid and $f : X \rightarrow B$ be a regular injective S -poset map. Then*

- (i) *For every $b \in B$, the sub S -poset $f^{-1}(b)$ of X is regular injective object in $\mathbf{Pos}\text{-}S$, so it is a complete poset.*
- (ii) *For $x \in X$ and $b \in B$ with $f(x) \leq b$, $\{x' \in f^{-1}(b) \mid x \leq x'\}$ has a minimum element x_b (also we have the dual of this fact).*

Proof. Applying Proposition 3.4 and the above proposition we get the result. □

Now, we consider the category $\mathbf{Pos}\text{-}S$ as a sub category of \mathbf{Cat} . On the other words, every S -poset is a category as a poset and all action-preserving monotone maps are functors. So we get the following result.

Proposition 3.7. *Every regular injective object in $\mathbf{Pos}\text{-}S/B$ is a topological S -poset map, considered as a functor.*

Proof. First by Corollary 3.6 and Theorem 3.1, one conclude that every regular injective object in $\mathbf{Pos}\text{-}S/B$ is a regular injective object in \mathbf{Pos}/B . Then part (3) from Example 2.3, says $(Emb)^\square = Top$, as we required. \square

Next, we try to prove the converse of this fact.

Proposition 3.8. *The functor $G_B : \mathbf{Pos}/B \rightarrow \mathbf{Pos}\text{-}S/B$ that equips monotone map in \mathbf{Pos}/B with the trivial action and so a map in $\mathbf{Pos}\text{-}S/B$, has a left adjoint.*

Proof. Define the functor $H_B : \mathbf{Pos}\text{-}S/B \rightarrow \mathbf{Pos}/B$ given by $H_B(h) = \bar{h} : A/\theta \rightarrow B$ with $\bar{h}([a]) = h(a)$ for every $h \in \mathbf{Pos}\text{-}S/B$. Where the poset A/θ was introduced in Theorem 12 from [5] and \bar{h} is a monotone map. If $g : A \rightarrow C$ is an S -poset map over B , then $H_B(g) : A/\theta \rightarrow C/\theta$ defined by $g([a]) = [g(a)]$ is a well-defined monotone map over B . The unit of this adjunction

$$\begin{array}{ccc} A & \xrightarrow{\eta_f} & A/\theta \\ & \searrow f & \swarrow G_B H_B(f) \\ & B & \end{array}$$

for an object $f : A \rightarrow B$ in $\mathbf{Pos}\text{-}S/B$, is the natural S -poset over B . It is a universal arrow to G_B because for a given S -poset map

$$\begin{array}{ccc} A & \xrightarrow{h} & P \\ & \searrow f & \swarrow G_B(l) \\ & B & \end{array}$$

where $l : P \rightarrow B$ is a monotone map, we have a unique S -poset map \bar{h} as the following diagram

$$\begin{array}{ccc} A/\theta & \xrightarrow{\bar{h}} & P \\ & \searrow G_B H_B(f) & \swarrow G_B(l) \\ & B & \end{array}$$

given by $\bar{h}([a]) = h(a)$. By similar proof in Theorem 12 from [5] one can prove that \bar{h} is a well defined S -poset map. The above diagram is commutative, since for every $[a] \in A/\theta$ we have:

$$G_B(l)(\bar{h}[a]) = G_B(l)(h(a)) = f(a) = \bar{f}[a] = H_B(f) = G_B H_B(f). \quad \square$$

We state the following result from [4] that will be used in the sequel.

Lemma 3.9. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be two functors such that $F \dashv G$. Also, let \mathcal{M} and \mathcal{N} be certain subclasses of \mathcal{C} and \mathcal{D} , respectively. If for all $f \in \mathcal{M}$, $Ff \in \mathcal{N}$, then for any \mathcal{N} -injective object D of \mathcal{D} , GD is an \mathcal{M} -injective object of \mathcal{C} .*

Theorem 3.10. *Let S be a pogroup. Then all topological S -poset functors are regular injective as S -poset map with trivial action.*

Proof. By similar proof of the Theorem 4.6 in [7], we can show that the functor $H_B : \mathbf{Pos}\text{-}S/B \rightarrow \mathbf{Pos}/B$ preserves order-embeddings, these are the regular monomorphisms in two category $\mathbf{Pos}\text{-}S/B$ and $\mathbf{Pos}\text{-}S$. Therefore, by Lemma 3.9 and the adjunction Proposition 3.8, the functor $G_B : \mathbf{Pos}/B \rightarrow \mathbf{Pos}\text{-}S/B$ preserves regular injective objects. Since, part (3) from the Example 2.3 says $(Emb)^\square = Top$ so we get the result. \square

Corollary 3.11. *Let S be a pogroup. Then (Emb, Top) is a weak factorization system for S -posets with trivial action.*

Proof. Every S -poset map $f : X \rightarrow B$ has a (Emb, Top) -factorization $f : X \rightarrow Y \rightarrow B$ as a monotone map in \mathbf{Pos} (see Corollary 2.7 in [1]) where Y is an S -poset with trivial action. \square

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